

Simple Interest

$I = Prt$, where P = principal amount invested
 r = annual rate of interest (in decimal form)
 t = time in years (could be portions of years)

Compound Interest

$$A = P\left(1 + \frac{r}{m}\right)^{nm} \quad \text{where } A = \text{compound amount}$$

p = principal
 r = annual rate of interest (in decimal form)
 n = number of years
 m = number of times per year compounded per year

Continuous Compounding

As can be noted by trying examples, compounding quarterly yields a larger result than compounding annually. Similarly, compounding monthly yields a larger result than compounding quarterly. In the same manner, compounding daily yields a larger result than compounding monthly. As the compounding takes place more and more frequently, the results increase only slightly. In fact, as these compounding intervals get shorter and shorter, the results get closer and closer to the results called continuous compounding given by the formula

$$A = Pe^{rt} \quad \text{where } A = \text{compound amount}$$

P = principal
 r = annual rate (in decimal form)
 t = time in years (can be parts of years; e.g. 3.4 years)
 e is the basis for natural logarithms.

Note the results of e^{rt} can easily be obtained on a scientific calculator.

Effective Annual Rate

The effective annual rate corresponding to a stated annual rate r per year compounded m times per year is given by the

formula $\left(1 + \frac{r}{m}\right)^m - 1$

An annuity with payments made at the end of each of m consecutive periods per year and compounding done on the same schedule at a stated annual rate r is called an ordinary annuity.

Future Value

The future value A of an ordinary annuity of m payments per year of R dollars each for n years with an annual rate of

interest r is
$$A = R \left[\frac{\left(1 + \frac{r}{m}\right)^{nm} - 1}{\frac{r}{m}} \right]$$

Sometimes you know you are going to need a certain amount in the future (maybe to make some large purchase) and you want to accumulate this amount by making periodic payments into an ordinary annuity. The necessary payment is given by

the following formula which is really the future value formula solved for the payment amount: $R = A \left[\frac{\frac{r}{m}}{\left(1 + \frac{r}{m}\right)^{nm} - 1} \right]$.

Sometimes such a fund is called a sinking fund.

The next formula is called the present value of an ordinary annuity formula. It is the amount of a one time payment which would grow (using compound interest) to the same amount as the accumulated payments would grow to in the ordinary annuity.

$P = R \left[\frac{1 - \left(1 + \frac{r}{m}\right)^{-nm}}{\frac{r}{m}} \right]$. Two of its uses are: a.) to find the amount of principal left to pay on a mortgage with some

number of payments remaining to be paid and b.) find what amount of money is needed in an account in order to make a certain number of payments of R size on a scheduled basis if the amount of money is earning interest compounded on the same schedule as the payments.

The last of the formulas is the previous one solved for R. It is used to find the amount of a mortgage payment where the mortgage amount is P and the payments are m times per year for n years and the annual rate of interest is r.

$$R = P \left[\frac{\frac{r}{m}}{1 - \left(1 + \frac{r}{m}\right)^{-nm}} \right]$$

Simple Interest

Example 1. Find the simple interest on a deposit of \$1250 for 5 years at 9% annual interest.

Use $I = Prt$ with $P = 1250$, $r = 0.09$ and $t = 5$. $I = 1250 \times 0.09 \times 5 = \562.50 .

Example 2. \$7,500 is deposited for 6 months in an account earning simple interest at an annual interest rate of 3.6%. What is the balance of the account at the end of the six months?

Use $I = Prt$ with $P = 7500$, $r = 0.036$ and $t = 0.5$. $I = 7500 \times 0.036 \times 0.5 = \135 . The balance after six months is the principal plus the interest; i.e., $P + I = \$7,500 + \$135 = \$7,635$.

Compound Interest

Example 3.a. Find the amount in an account after 5 years, based on an initial deposit of \$450 earning 9% annual interest compounded monthly.

Use $A = P \left(1 + \frac{r}{m}\right)^{nm}$ with $P = 450$, $r = 0.09$, $m = 12$ and $n = 5$. $A = 450 \left(1 + \frac{0.09}{12}\right)^{5 \times 12} = \704.56 (to the nearest penny).

Example 3 b. How much interest is earned in Example 3a?

Subtract the principal from the amount present after 5 years. The difference is the interest. $\$704.56 - \$450 = \$254.56$.

Continuous Compounding

Example 4.a Find the amount in an account after 5 years, if the initial deposit is \$450 and the account is earning 9% compounded continuously.

Use $A = Pe^{rt}$ with $P = \$450$, $r = 0.09$, and $t = 5$. $A = 450e^{0.09(5)} = 450e^{0.45} = \705.74

Example 4. b. How much interest is earned in Example 4.a?.

Subtract the principal from the amount present after 5 years. The difference is the interest. $\$705.74 - \$450 = \$255.74$.

Effective Annual Interest

Example 5. The interest-bearing checking account at one bank earns 6.2% annual interest compounded quarterly (4 times per year). The corresponding account at a second bank earns 6.17% annual interest compounded monthly. Which bank's account is better?

Compare the effective annual interest $\left(1 + \frac{r}{m}\right)^m - 1$ for the two accounts. For the first bank, it is $\left(1 + \frac{0.062}{4}\right)^4 - 1 = 0.063456$ or 6.3456% effective annual interest. For the second bank, it is $\left(1 + \frac{0.0617}{12}\right)^{12} - 1 = 0.063475$ or 6.3475% effective annual interest. Therefore, the account at the second bank is (slightly) better.

Future Value

(1) ANNUITY

Example 6. Suppose Joanne deposits \$100 per month into her retirement fund for 30 years. The fund earns 7.2% annual interest compounded monthly. What is the value of her fund after 30 years? How much of that value did she deposit and how much is interest?

Use the future value formula $A = R \left[\frac{\left(1 + \frac{r}{m}\right)^{nm} - 1}{\frac{r}{m}} \right]$ with $R = 100$, $r = 0.072$, $m = 12$ and $n = 30$. So, $A = 100 \left[\frac{\left(1 + \frac{0.072}{12}\right)^{30 \times 12} - 1}{\frac{0.072}{12}} \right] = \$126,922.54$ (to the nearest penny). Over the 30 years, Joanne makes $30 \times 12 = 360$ monthly

deposits of \$100 for a total of \$36,000. The balance, $\$126,922.54 - \$36,000 = \$90,922.54$, is interest earned.

Example 7. Mary invests \$100 per month into an account earning 6% annual interest compounded monthly. She does this from age 25 to age 35 and stops (a period of 10 years). The balance in the account at that point then is allowed to accumulate interest until she reaches age 65. On the other hand, Laurel begins, at age 35, investing \$100 per month into an account earning 6% annual interest compounded monthly. She continues until she reaches age 65 (a period of 30 years). How much money has each accumulated by age 65 and how much money did each deposit?

Mary: For the first stage (age 25 to 35), use the future value formula with $R = 100$, $r = 0.06$, $m = 12$ and $n = 10$.

$$A = 100 \left[\frac{\left(1 + \frac{0.06}{12}\right)^{10 \times 12} - 1}{\frac{0.06}{12}} \right] = \$16,387.93 \text{ (Mary's balance at age 35)}$$

For the second stage (age 35 to 65), the above balance accumulates compound interest. Apply the compound interest formula with $P = 16387.93$, $r = 0.06$, $m = 12$ and $n = 30$.

$$A = 16387.93 \left(1 + \frac{0.06}{12}\right)^{30 \times 12} = \$98,697.54 \text{ (Mary's balance at age 65)}$$

Mary made 120 monthly payments of \$100 for a total of \$12,000.

Laurel: Use the future value formula with $R = 100$, $r = 0.06$, $m = 12$ and $n = 30$.

$$A = 100 \left[\frac{\left(1 + \frac{0.06}{12}\right)^{30 \times 12} - 1}{\frac{0.06}{12}} \right] = \$100,451.50 \text{ (Laurel's balance at age 65)}$$

Laurel made 360 monthly payments of \$100 for a total of \$36,000.

<u>Summary:</u>	<u>Total deposited</u>	<u>Balance at age 65</u>
Mary	\$12,000	\$ 98,697.54
Laurel	\$36,000	\$100,451.50

(2) SINKING FUND

Example 8. When the Wilsons' son was born, they were advised that they should save \$40,000 by his 18th birthday for college tuition. They decided to make quarterly payments into a trust fund which earns 10.8% interest compounded quarterly. How much was each payment?

Use $R = A \left[\frac{\frac{r}{m}}{\left(1 + \frac{r}{m}\right)^{nm} - 1} \right]$ with $A = 40,000$, $m = 4$, $n = 18$ and $r = 0.108$. This gives

$$R = 40000 \left[\frac{\frac{0.108}{4}}{\left(1 + \frac{0.108}{4}\right)^{18 \times 4} - 1} \right] = \$185.92. \text{ Note that during the course of the 18 years, they made 72 payments of}$$

\$185.92 for a total of \$13,386.24.

Example 9. Redo the previous example using monthly payments and compounding.

Change m to 12 in the above calculation. $R = 40000 \left[\frac{\frac{0.108}{12}}{\left(1 + \frac{0.108}{12}\right)^{18 \times 12} - 1} \right] = \$60.75.$ This time, over the 18 years, they

made 216 payments of \$60.75 for a total of \$13,122.00.

Present Value

(1) ANNUITY

Example 10. Mr. Wilson plans to retire on his 65th birthday. His retirement account earns 9.6% annual interest compounded monthly. His goal is to receive a monthly payment of \$2,500 from the retirement account for a period of 25 years. How much should be present in the account when he retires?

Use $P = R \left[\frac{1 - \left(1 + \frac{r}{m}\right)^{-nm}}{\frac{r}{m}} \right]$ with $R = 2500$, $r = 0.096$, $m = 12$ and $n = 25$. This gives

$$P = 2500 \left[\frac{1 - \left(1 + \frac{0.096}{12}\right)^{-25 \times 12}}{\frac{0.096}{12}} \right] = \$283,878.63.$$

(2) LOAN PAYMENT / MORTGAGE

Example 11. Suppose that you buy a house for \$90,000 with a 30 year mortgage which accrues 7.02% annual interest compounded monthly. What will the mortgage payment be? How much will you ultimately pay for the house and how much of this is interest?

Use $R = P \left[\frac{\frac{r}{m}}{1 - \left(1 + \frac{r}{m}\right)^{-nm}} \right]$ with $P = 90000$, $r = 0.0702$, $m = 12$ and $n = 30$. This gives

$$R = 90000 \left[\frac{\frac{0.0702}{12}}{1 - \left(1 + \frac{0.0702}{12}\right)^{-30 \times 12}} \right] = \$599.98. \text{ You will make } 30 \times 12 = 360 \text{ payments for a total of } 360 \times \$599.98 =$$

\$215,992.80. The difference between this and the purchase price is interest: $\$215,992.80 - \$90,000 = \$125,992.80$ interest.

Example 12. Redo Example 11 using a 20 year mortgage.

Change n to 20. $R = 90000 \left[\frac{\frac{0.0702}{12}}{1 - \left(1 + \frac{0.0702}{12}\right)^{-20 \times 12}} \right] = \$698.85. \text{ You will make } 20 \times 12 = 240 \text{ payments for a total of } 240$

$\times \$698.85 = \$167,724.00$. The interest is $\$167,724.00 - \$90,000 = \$77,724$. Note the differences between Examples 11 and 12. By paying slightly less than \$100 more per month, you pay off the mortgage 10 years sooner and save $\$215,992.80 - \$167,724.00 = \$48,268.80$!!

(3) CURRENT BALANCE ON A LOAN

Example 13. Consider the scenario in Example 11. How much is owed on the house (i.e., what is the balance on the mortgage) after 18 years of payments.

12 years remain on the mortgage, so use $P = R \left[\frac{1 - \left(1 + \frac{r}{m}\right)^{-nm}}{\frac{r}{m}} \right]$ with $R = 599.98$, $r = 0.0702$, $m = 12$ and $n = 12$. This

gives

$$P = 599.98 \left[\frac{1 - \left(1 + \frac{0.0702}{12}\right)^{-12 \times 12}}{\frac{0.0702}{12}} \right] = \$58,821.75.$$

Homework.

1. a. Find the amount of simple interest on a deposit of \$10000 for 8 years at 6% interest rate.
b. How much would you have including interest at the end of the 8 years?
2. a. Find the amount in the account in #1 if compound interest is used(compounded annually).
b. How much interest will be earned?
c. Find the amount in the account in #1 if continuous compounding is used.
d. How much interest will be earned?
3. a. What is the effective annual rate for an account which pays 6.1% compounded quarterly?
b. What is the effective annual rate for an account which pays 6.05% compounded daily?
4. You deposit \$150 per month for 20 years into an ordinary annuity with annual rate of interest 9%. How much will be in the account after 20 years?
5. You know that 10 years from now you will need to replace a piece of equipment at an expected cost of \$30000. You want to create a sinking fund by making quarterly payments into a mutual fund account earning 10% on an annual basis. How much will the quarterly payments have to be?
6. a. What is the monthly payment for a 30 year mortgage of \$100,000 at 8.25% annual rate?
b. What is the monthly payment for a 15 year mortgage of \$100,000 at 7.35% annual rate?
7. In both 6a. and 6b., how much interest is paid if the contracts are met on time?
8. In 6a., how much principal is owed after making payments on time for 10 years?
9. You want to set up an annuity that will pay out \$1000 per month for 12 years. The annual rate of interest is 9%. How much should be put in the account?
10. For ten years Sally makes \$200 monthly payments to a fund earning 11% annual rate of interest compounded monthly. At the end of ten years she stops making payments, but leaves the accumulated amount in the fund for twenty years earning interest at same rate as before. How much will she have at the end of this twenty year period? At the time Sally discontinues making payments, Jim begins to make \$200 monthly payments into the same fund. How much will he have in the account at the end of twenty years?